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BOUNDED REALIZATION OF ASYMPTOTICALLY ATTAINABLE ELEMENTS†

S. I. MORINA and A. G. CHENTSOV

Ekaterinburg

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Problems of the structure of asymptotically attainable elements (AAE) generated by the action of "control functions" that satisfy generally non-convex constraints of a functional character to a high degree of accuracy are considered. No resource type conditions are assumed, which leads to an "unbounded" formulation of the problem concerned with the asymptotic behaviour of attainability domains and their abstract analogues. Necessary and sufficient conditions for an exhaustive realization of the AAE in the class of integrally bounded approximate solutions are established in terms of the generalized problem in the class of finitely additive vector-valued measures.

1. FORMULATION OF THE PROBLEM

We shall study the attainability domain of the linear controlled system

$$\dot{x}(t)=A(t)x(t)+B(t)f(t),\quad x(t_0)=x_0,\quad t_0\leqslant t\leqslant \theta_0 \tag{1.1}$$

Let the phase space of (1.1) have dimension n. As in [1–3], we consider the attainability domain for the first k coordinates at the time θ_0 ($k\leqslant n$) to be the set of all finite states generated by the set of vector-valued control programs $f=(f_1,\ldots,f_r)$, where r is a positive integer, such that the condition

$$\int_{t_0}^{\theta_0} S(t)f(t)dt\in \mathbf{Y} \tag{1.2}$$

is satisfied. Here $A(\cdot)$ is a component-wise continuous $(n\times n)$ matrix-valued function on $[t_0,\theta_0]$, each component of the $(n\times r)$ matrix-valued function $B(\cdot)$ defined in $[t_0,\theta_0]$ being obtained by joining together a finite number of restrictions (to intervals of the form $[\alpha,\beta[$, where $t_0\leqslant\alpha<\beta\leqslant\theta_0$) of continuous functions in $[t_0,\theta_0]$. The structure of the matrix-valued function $S(\cdot)$ is assumed to be the same as in the case of $B(\cdot)$, the "dimensions" of $S(\cdot)$ being $m\times r$. $\mathbf{Y}$ is a closed set in the m-dimensional space $\mathbf{R}^m$. We shall assume that all components of the control program $f=(f_1,\ldots,f_r)$ are piecewise-constant, right-continuous, and non-negative.

Let F be the set of all such control programs $f=(f_1,\ldots,f_r)$ (with piecewise-constant, right-continuous, and non-negative components). If $f\in F$, then we denote by $x_f(\cdot)=(x_f(t),t_0\leqslant t\leqslant\theta_0)$ the trajectory of (1.1) generated by f and starting from an initial position (t_0,x_0). Let π be the projection from $\mathbf{R}^n$ onto $\mathbf{R}^k$ which makes any n-dimensional vector correspond to the vector formed by the first k coordinates. Then, along with the points $\pi(x_f(\theta_0))$ forming directly the attainability domain under the condition (1.2), it makes sense to consider their limits, i.e. the limits of the sequences

$$p\mapsto\pi(x_{f_p}(\theta_0)):\quad N\to\mathbf{R}^k \tag{1.3}$$

corresponding to approximate solutions $(f_p)_1^\infty$ in F that satisfy each weakened version of condition (1.2) (with an ε-neighbourhood on the right-hand side of (1.2), $\varepsilon>0$) starting from some instant of time. If $z_*\in\mathbf{R}^k$ satisfies the above asymptotic attainability condition, the question arises of the properties of the approximate solutions $(f_p)_1^\infty$ that realize z_* as the limit of (1.3). Below we discuss the conditions of a ("bounded") realization of z_*, under which each version of the solution sequence $(f_p)_1^\infty$ satisfying condition (1.3) of convergence to z_* (which also satisfies (1.2) to within ε for almost all $p\in N$ when $\varepsilon>0$) has a bounded energy input sequence.

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In this connection we consider a simple example of the problem of the control of a point mass on a straight line. In this case (1.1) has the form ($n = 2, k = 1$)

$$\dot{x}_1(t) = x_2, \quad \dot{x}_2(t) = f(t) \tag{1.4}$$

with zero initial conditions $x_{01} = x_{02} = 0$; $t_0 \triangleq 0$, $\theta_0 \triangleq 1$.

As (1.2), we shall use the condition $x_{f,1}(1) = 1/2$ (its reduction to (1.2) is obvious and will be omitted). There is a (scalar) control function $f \in F$ for which the above condition is satisfied accurately (for example, the control function that is identically equal to one). It follows that the point $z_* = 1/2$ admits of an asymptotic realization in the class of integrally bounded approximate solutions (in this case the solution sequence $(f_p)_1^\infty$ can be chosen to be stationary). On the other hand, there is also an "unbounded" asymptotic realization of the same point z_*.

For if $\delta \in]0, 1[$, we define $f^{(\delta)} \in F$ to be the function that is non-zero only over the interval $[1 - \delta, 1[$ and which takes the constant value δ^{-2} in this interval. If the sequence of approximate solutions $(f_p)_1^\infty$ is now taken to be

$$f_q = f^{(\delta)}\Big|_{\delta=(2q)^{-1}} \quad (q \in N)$$

then, in particular, we shall obtain an unbounded asymptotic realization of z_*. Of course, in the specific case in question we are dealing with infinitely many versions of an accurate realization of z_*, which implies, in particular, that the realization is asymptotic. Nevertheless, even this rather schematic example demonstrates that, for any point chosen by the investigator, different ways for its asymptotic realization on the terminal states of (1.1) may exist simultaneously in principle. In many cases the asymptotic realization of z_* is always "bounded", so that the property of asymptotic attainability of the given point is of better quality in some sense.

For if we consider the construction of the asymptotic attainability set [4] in the whole phase space under the same condition $x_{f,1}(1) = 1/2$, then, for any planar vector z_* chosen in this set, every asymptotic realization of z_* will turn out to be "bounded" (on account of the velocity coordinate). The condition $x_{f,1}(1) = 1/2$ itself can be replaced by a more general one (see (1.2)), but the assertion on the nature of the asymptotic realization will remain unchanged. Regrading the aforesaid "direct" condition, we observe that here all possible points $(1/2\,;\,\upsilon)$, where $\upsilon \geqslant 1/2$, are asymptotically attainable.

We will consider one more similar example for the following elementary controlled system ($n = 1, r = 1$)

$$\dot{x}(t) = f(t), \quad 0 \leqslant t \leqslant 1$$

The restrictions on the choice of the control have the form

$$\int_0^1 tf(t)dt \leqslant 0, \quad f \in F \tag{1.5}$$

Suppose that the functional

$$f \mapsto g\left(\int_0^1 f(t)dt\right): \quad F \to \mathbf{R}$$

$$g: \ \mathbf{R} \to \mathbf{R}, \quad g(x) = \begin{cases} 2|x - 1/2|, & -\infty \leqslant x \leqslant 1 \\ 1/x, & 1 \leqslant x \leqslant +\infty \end{cases}$$

is given. We consider the point $z^* = 0$.

On the one hand, this point can be realized as the limit of the sequence

$$\left(g\left(\int_0^1 f_n(t)dt\right)\right)_1^\infty, \quad f_n(t) = \begin{cases} n/2 & 0 \leqslant t < 1/n \\ 0, & 1/n \leqslant x < 1 \end{cases}, \quad \forall n \in N$$

It is obvious that (1.5) is satisfied for the control f_n (for almost all $n \in N$) apart from $\varepsilon > 0$ and

$$\forall n \in N: \ \int_0^1 f_n(t)dt = \frac{1}{2}$$

It follows that $(f_n)_1^\infty$ is an integrally bounded approximate solution, which realizes $z_* = 0$ in the limit (in fact, accurately) on the values of g.

On the other hand, consider the sequence

$$(h_n)_1^\infty: \ h_n(t) = \begin{cases} n^3, & 0 \leqslant t < 1/n^2 \\ 0, & 1/n^2 \leqslant t < 1 \end{cases}$$

Obviously, $(h_n)_1^\infty$ is an approximate solution realizing the same point $z_* = 0$ on the values of g, but it is not integrally bounded. Here it is taken into account that

$$\int_0^1 th_n(t)dt = \int_0^{1/n^2} n^3 t dt = \frac{1}{2n} \to 0$$

The simple examples considered above indicate that an asymptotically attainable element z_* may admit of both a bounded and an unbounded realization. It is of interest to establish conditions which exclude the "unbounded" version of the asymptotic realization of z_*. The present paper is devoted to a study of such conditions. Note that the properties that are the basics of the phenomenon under investigation are present not only in control problems. Another important example of this kind can be obtained by considering restrictions on the form on inequalities, which are typical of mathematical programming problems. There are also other realizations of the general construction considered below, in which no specific features of control problems are used. A "systems operator" with values in a topological space is considered in these general constructions.

An extension of this kind becomes necessary even in problems concerned with the control of system (1.1), (1.2) if the asymptotic behaviour of trajectories is considered, rather than just their terminal values. In this case a vector-valued function, which may, in general, be discontinuous, plays the role of z_*. It proves useful to employ the pointwise-convergence topology to describe the passage to the limit. In this connection, we shall henceforth consider a special topological construction, which enables us to study from a general standpoint the issue of the asymptotic realization, invoking the notion of Moore–Smith convergence if necessary.

2. GENERAL DEFINITIONS AND NOTATION

We shall use topology and finitely additive measure theory, which are required to study the applied problems in question. Henceforth we shall use quantifiers and predicates. We shall use the following convention. The expression $\forall_X S[X \neq \emptyset]$ ($\exists_X S[X \neq \emptyset]$) will be employed as a brief notation of the following statement: for every set (there exists a set) X such that $X \neq \emptyset$. We denote the real axis by $\mathbf{R}$; $N = \{1, 2, \dots\}$. We put $\forall m \in N$: $\overline{1,m} \triangleq \{i \in N \mid i \leqslant m\}$. If A and B are sets, then we denote by B^A the set of all functions f from A to B. If $f \in B^A$ and C is a subset of A, we put $f^1(C) = \{f(x): x \in C\}$. If (X, τ) is a topological space (TS) and A a subset of X, then we denote by $\mathrm{cl}(A, \tau)$ the closure of A in (X, τ). If (X, τ) is a TS and $x \in X$, then $\mathbf{N}_\tau(x)$ will denote the family of all neighbourhoods [5] of the point x.

Below we use generalized sequences or directions [5] in a TS to represent a passage to the limit (a sequence being a special case of a direction). If T is a set, then we denote by DIR[T] the set of all directions [5] on T. For $\angle \in \mathrm{DIR}[T]$ we call the pair $(T, \angle)$ a directed set (DS), as usual. Furthermore, if H is a set and $h \in H^T$, then the triple $(T, \angle, h)$ is called a direction in H. If $(T, \angle)$ is a DS, then we denote by $(\angle - \mathrm{conf})[T]$ the family of all sets $P \subset T$ such that $\forall x \in T\, \exists y \in P$: $x \angle y$. If (X, τ) is a TS and $(T, \angle, h)$ is a direction in X, we put

$$(\tau - \mathrm{cl})[T;\angle;h] \triangleq \{x \in X \mid \forall Q \in \mathbf{N}_\tau(x):\ h^{-1}(Q) \in (\angle - \mathrm{conf})[T]\}$$

(the set of all limiting points of $(T, \angle, h)$). If (X, τ) is a TS and $(T, \angle, h)$ is a direction in X, we shall write $(T; \angle; h] \xrightarrow{\tau} x$ if $(T, \angle, h)$ converges to x in (X, τ). If $m \in N$ and H is a non-empty set, we get $H^m \triangleq H^{\overline{1,m}}$. If (T, τ) with $T \neq \emptyset$ is a TS and $m \in N$, then we denote by $\otimes^m[\tau]$ the natural topology in $T^m = T \times \dots \times T$ (m times) corresponding to the product of m copies of (T, τ). Linear operations, multiplication, and order will be defined in a pointwise manner.

We fix a non-empty set E and a semi-algebra [4, 6] $\mathbf{L}$ of subsets of E. We denote by $(\mathrm{add})_+[\mathbf{L}]$ the cone of all non-negative real-valued finitely additive measures on $\mathbf{L}$ and by $\mathbf{A}[\mathbf{L}]$ the linear subspace $\mathbf{R}^\mathbf{L}$ generated by the cone. Next, we fix $\eta \in (\mathrm{add})_+[\mathbf{L}]$. We follow the definitions and notation of [7–9] in respect of the elements of finitely additive measure and integration theory. We denote by $B_0(E, \mathbf{L})$ the set of all step functionals on E in the sense of $(E, \mathbf{L})$ (only functions with a finite set of values will be considered as step functions). A simple example of this space, related to the natural pointer semialgebra of the interval $[t_0, \theta_0[$, is provided by the set of all piecewise-constant right-continuous functions on $[t_0, \theta_0[$.

Turning to the general case, we introduce the positive cone $B_0^+(E, \mathbf{L})$ in $B_0(E, \mathbf{L})$, the elements of which play a role similar to that of the components of $f \in F$ in Section 1. We will denote by $B(E, \mathbf{L})$

the closure of $B_0(E, \mathbf{L})$ in the space $\mathbf{B}(E)$ of all bounded functionals on E equipped with the supremum norm $\|\cdot\|$ ($B(E, \mathbf{L})$ consists of uniform limits of convergent sequences in $B_0(E, \mathbf{L})$ and only such limits). If $\mathbf{L}$ is a σ-algebra of subsets of E, then ($B(E, \mathbf{L})$ is the set of all $\mathbf{L}$-measurable functionals in $\mathbf{B}(E)$. Then $B(E, \mathbf{L})$, being a closed subspace in $\mathbf{B}(E)$, $\|\cdot\|$, is a Banach space with positive cone $B^+(E, \mathbf{L})$. We denote by $B^*(E, \mathbf{L})$ the dual topological space to $B(E, \mathbf{L})$. It is fully characterized by the measures of $\mathbf{A}(\mathbf{L})$ in the sense that even the simplest construction of the integral with respect to a finitely additive measure [8] defines an isometric isomorphism $\mathbf{A}(\mathbf{L})$ on $B^*(E, \mathbf{L})$, associating a functional from $B(E, \mathbf{L})$ with each measure $\mu \in \mathbf{A}(\mathbf{L})$, the values of the functional being the μ-integrals of functionals from $B(E, \mathbf{L})$.

The duality ($B(E, \mathbf{L}), \mathbf{A}(\mathbf{L})$ defines the *-weak topology $\tau_*(\mathbf{L})$ in $\mathbf{A}(\mathbf{L})$ as the weakest topology in $\mathbf{A}(\mathbf{L})$ in which the integral of every function from $B(E, \mathbf{L})$ is continuous with respect to the finitely additive measure (of bounded variation) appearing in the integral. We know that $(\mathbf{A}(\mathbf{L}), \tau_*(\mathbf{L}))$ is a locally convex space, in which the compactness conditions are given by Alaoghu's theorem [10].

If $f \in B(E, \mathbf{L})$, then we denote by $f * \eta$ the indefinite η-integral of f [8]. We set $(\mathrm{add})^+[\mathbf{L}; \eta] \triangleq \{\mu \in (\mathrm{add})_+[\mathbf{L}] \mid \forall L \in \mathbf{L}: (\eta(L) = 0) \Rightarrow (\mu(L) = 0)\}$. We denote by $\tau^*_{\eta(\mathbf{L})}$ the (relative) topology $(\mathrm{add})^+[\mathbf{L}; \tau]$ induced from $((\mathbf{A}(\mathbf{L}), \tau_*(\mathbf{L}))$; $(\mathrm{add})^+{}_r[\mathbf{L}; \eta] \triangleq (\mathrm{add})^+[\mathbf{L}; \eta]^r$;

$$(\mathrm{add})^+_r[\mathbf{L};\eta] \triangleq (\mathrm{add})^+[\mathbf{L};\eta]^r;\ (\mathrm{add})^+_r[\mathbf{L};\eta], \otimes^r[\tau^*_\eta(\mathbf{L})]) \tag{2.1}$$

is the locally compact space of generalized control functions admitting of an everywhere dense embedding $B^+_{0,r}[E, \mathbf{L}] \triangleq B^+_0(E; \mathbf{L})^r$ by the operator $\mathbf{I}$ defined as

$$(f_i)_{i\in\overline{1,r}} \mapsto (f_i * \eta)_{i\in\overline{1,r}}:\ B^+_{0,r}[E;\mathbf{L}] \to (\mathrm{add})^+_r[\mathbf{L};\eta]$$

Namely [7], $(\mathrm{add})^+{}_r[\mathbf{L}; \eta] = \mathrm{cl}(\mathbf{I}^1(B^+_{0,r}[E; \mathbf{L}), \otimes^r[\tau^*{}_\eta(\mathbf{L})])$. Let $\forall_Q S[Q \neq \emptyset]\ \forall \angle \in (\mathrm{DIR})[Q]$

$$\mathbf{B}_r(Q, \angle, E, \mathbf{L}, \eta) \triangleq \{g \in B^+_{0,r}[E;\mathbf{L}]^Q \mid \exists d \in Q\ \exists c \in]0, \infty[$$

$$\forall q \in Q:\ (d \angle q) \Rightarrow \left(\sum_{i=1}^{r} \int_E g(q)(i) d\eta \leqslant c\right)\Bigg\} \tag{2.2}$$

3. CONDITIONS FOR A BOUNDED REALIZATION OF ASYMPTOTICALLY ATTAINABLE ELEMENTS

Let $\mathbf{F}$ be a non-empty family of subsets of $B^+_{0,r}[E; \mathbf{L}]$ such that $\forall A \in \mathbf{F}\ \forall B \in \mathbf{F}\ \exists C \in \mathbf{F}: C \subset A \cap B$. It will play the role of asymptotic constraints. Furthermore, let (θ, ϑ) with $\theta \neq \emptyset$ be a given TS. We fix an operator

$$w:\ (\mathrm{add})^+_r[\mathbf{L}; \eta] \to \theta \tag{3.1}$$

continuous in the sense of the TS (2.1) and (θ, ϑ). Moreover, we set $W \triangleq w \circ \mathbf{I}$, so that W is the operator

$$(f_i)_{i\in\overline{1,r}} \mapsto w((f_i * \eta)_{i\in\overline{1,r}}):\ B^+_{0,r}[E;\mathbf{L}] \to \theta \tag{3.2}$$

There are many specific classes of problems admitting of the representation (3.1), (3.2). In particular, this is the case for the problem concerned with the study of the asymptotic behaviour of attainability domains considered in Section 1 [4, Chap. VI; 7, 9]. The construction of the operator (3.1) for this case involves an extension of Cauchy's formula [11, 12].

In turn, this and some other applications are related to a representation which can be essentially characterized by the following class of operators W

$$W(f) = g\left(\int_E H(x) f(x) \eta(dx)\right),\quad f \in B^+_{0,r}[E;\mathbf{L}] \tag{3.3}$$

where g is a continuous vector-valued function in an approximate finite-dimensional space. In the case (3.3) the operator W can be constructed using the notion of an indefinite integral.

Let $\underset{T}{\forall S}[T \neq \emptyset] \quad \forall \preccurlyeq \in (\mathrm{DIR})[T] \quad \forall \omega \in \theta$:

$$AS[T, \preccurlyeq, \omega] \triangleq \{h \in B_{0,r}^{+}[E;\mathbf{L}]^{T} \mid (\forall U \in \mathbf{F} \;\; \exists m \in T$$

$$\forall t \in T\colon \;\; (m \preccurlyeq t) \Rightarrow (h(t) \in U)) \& ((T, \preccurlyeq, W \circ h) \overset{\vartheta}{\rightarrow} \omega)\}$$

$$\mathbf{AC} \triangleq \{\omega \in \theta \mid \underset{T}{\exists S}[T \neq \emptyset] \;\; \exists \preccurlyeq \in \mathrm{DIR}[T]\colon \;\; \mathrm{AS}[T, \preccurlyeq, \omega] \neq \emptyset\} \tag{3.4}$$

$$\mathbf{BAC} \triangleq \{\omega \in \mathbf{AC} \mid \underset{T}{\forall S}[T \neq \emptyset] \;\; \forall \preccurlyeq \in \mathrm{DIR}[T]\colon$$

$$\mathrm{AS}[T, \preccurlyeq, \omega] \subset \mathbf{B}_r(T, \preccurlyeq, E, \mathbf{L}, \eta)\} \tag{3.5}$$

Theorem 1. Let $\omega \in \mathbf{AC}$. Then the following assertions are equivalent

1. $\omega \in \mathbf{BAC}$;
2. $(\underset{T}{\forall S}[T \neq \emptyset] \;\; \forall \preccurlyeq \in \mathrm{DIR}[T] \;\; \forall h \in \mathrm{AS}[T, \preccurlyeq, \omega]$:

$$(\otimes^r[\tau_\eta^*(\mathbf{L})] - \mathrm{cl})[T; \preccurlyeq; \mathbf{I} \circ h] \neq \emptyset).$$

Proof. The implication $1 \Rightarrow 2$ is a simple consequence of the compactness of

$$\Xi_r^+[c] \triangleq \left\{(\mu_i)_{i \in \overline{1,r}} \in (\mathrm{add})_r^+[\mathbf{L};\eta] \,\Big|\, \sum_{i=1}^{r} \mu_i(E) \leqslant c\right\} \quad (c \geqslant 0)$$

in the topological space (2.1) [4, Chaps IV, VI].

Suppose that ω satisfies condition (2). We shall select an approximate solution realizing ω in the limit. Thus, let $(\Lambda, \preccurlyeq, \varphi)$ be a direction in $B_{0,r}^{+}[E; \mathbf{L}]$ such that $\varphi \in \mathrm{AS}[\Lambda, \preccurlyeq, \varphi]$. Then $\varphi \in \mathbf{B}_r(\Lambda, \preccurlyeq, E, \mathbf{L}, \eta)$. This assertion can be proved by assuming the contrary and using the well-known construction of the directed product [5]. Another important feature is related to the following characteristic property of the limiting point of a direction in a topological space: the preimage of a neighbourhood by the operator defining the direction is a confinal set [5]. We recall that $(\Lambda, \preccurlyeq, \varphi)$, with $\varphi \in \mathrm{AS}[\Lambda, \preccurlyeq, \varphi]$ has been chosen arbitrarily so that $\omega \in \mathbf{BAC}$ by (3.5), i.e. the implication $1 \Rightarrow 2$ has been established.

4. A SPECIAL CASE

Everywhere below we shall consider the case of a metrizable [5] space (θ, ϑ). Let ρ be a metric generating the topology ϑ.

Condition 1.

$$\exists \omega^0 \in \theta \;\; \forall a \in \,]0, \infty[\;\; \exists b \in [0, \infty[$$

$$\forall \mu \in (\mathrm{add})_r^+[\mathbf{L};\eta]\colon \;\; \left(b < \sum_{i=1}^{r} \mu(i)(E)) \Rightarrow (a \leqslant \rho(\omega^0, w(\mu))\right)$$

Theorem 2. Let Condition 1 be satisfied. Then $\mathbf{AC} = \mathbf{BAC}$.

Proof. Let $\omega \in \mathbf{AC}$. Then one can find a direction $(T, \angle, h)$ in $B_{0,r}^{+}[E; \mathbf{L}]$ such that $h \in \mathrm{AS}[T, \angle, \omega]$. We choose any direction $(T, \angle, h)$ in $B_{0,r}^{+}[E; \mathbf{L}]$ having the aforesaid properties. Then, in particular, $\rho(W(h(t)), \omega) < 1$ starting from some instant of time. If ω^0 corresponds to Condition 1, then $\rho(W(h(t)), \omega^0) < \rho(\omega^0, \omega) + 1$ starting from some instant of time. By Condition 1, it can be shown that $b \in [0, \infty[$, so that $\forall \mu \in (\mathrm{add})_r^+[\mathbf{L}; \eta]$

$$\left(b < \sum_{i=1}^{r} \mu(i)(E)) \Rightarrow (\rho(\omega^0, \omega) + 1 \leqslant \rho(\omega^0, w(\mu))\right) \tag{4.1}$$

Along with this, $\forall\, t \in T$: $(W \circ h)(t) = w((\mathbf{I} \circ h)(t))$. Then $\rho(\omega^0, w((\mathbf{I} \circ h)(t))) < \rho(\omega^0, \omega) + 1$ starting from some instant of time, so that (see (4.1))

$$\sum_{i=1}^{r} \int h(t)(i)d\eta = \sum_{i=1}^{r} (\mathbf{I} \circ h)(t)(i)(E) \leq b$$

starting from some instant of time. In other words, $h \in \mathbf{B}_r(T, \angle, E, \mathbf{L}, \eta)$. Since the approximate solution $(T, \angle, h)$ (which realizes ω) has been chosen arbitrarily, we find that $\omega \in$ **BAC**. The inclusion **AC** $\subset$ **BAC** has been proved, which, by (3.5), concludes the whole proof.

Example. Let the following conditions be satisfied: (1) $p \in N$ and $(\theta, \vartheta) - \mathbf{R}^p$ with the ordinary topology of coordinate-wise convergence; (2) $\omega^0 \in \mathbf{R}^p$; (3) $(i,j) \mapsto M_{i,j}: \overline{1,p} \times \overline{1,r} \to B^+(E; \mathbf{L})$ is a matrix-valued function such that $\exists \alpha \in]0, \infty[\forall j \in \overline{1,r}\ \exists i \in \overline{1,p}\ \forall x \in E: \alpha \leq M_{ij}(x)$; (4) W can be represented as

$$w(f) \triangleq \omega^0 + \left(\sum_{j=1}^{r} \int_E M_{i,j} f(j) d\eta \right)_{i \in \overline{1,p}}, \quad f \in B_{0,r}^{+}[E;\mathbf{L}] \tag{4.2}$$

For W given by (4.2) one can realize (3.2) in terms of the following continuous operator w

$$W(\mu) \triangleq \omega^0 + \left(\sum_{j=1}^{r} \int_E M_{i,j} d\mu_j \right)_{i \in \overline{1,p}}, \quad \mu \in (\text{add})_r^{+}[\mathbf{L};\eta]$$

If ρ is now defined to be the supremum metric (i.e. the metric generated by the supremum norm is $\mathbf{R}^p$), then Condition 1 is satisfied. For example, the study of the asymptotic behaviour of attainability domains in the whole phase space for a vector point mass controlled by a force with non-negative components can be reduced to this form. (The choice of the control force as a programme is also subject to restrictions of the form (1.2); a family **F** (Section 3) of general form can of course also be used to impose "asymptotic" restrictions.)

The following system serves as another example satisfying Conditions 1–4

$$\dot{x}(t) = A(t)x + f(t), \quad t_0 \leq t \leq \theta, \quad x(t_0) = x_0, \quad x \in \mathbf{R}^r$$

where A is a diagonal $(r \times r)$-matrix with elements $\lambda_1 \in \mathbf{R}, \ldots, \lambda_r \in \mathbf{R}$, and f is an r-dimensional vector-valued control function.

In this case the role of the matrix-valued function M is played by the fundamental matrix of solutions of the corresponding homogeneous system. Let $\alpha_i \triangleq e^{-|\lambda_i|(\theta - t_0)}$, $i \in \overline{1,r}$, and $\alpha \triangleq \inf_{i \in \overline{1,r}}\{\alpha_i\}$. Then, obviously, $\forall i \in \overline{1,r}$ $\forall t \in [t_0, \theta]$: $\exp[\lambda_i(\theta - t)] \geq \alpha$.

Returning to the general case of a metrizable TS (θ, ϑ), we observe that if **F** has a fundamental sequence of sets [10], then in (3.4) and (3.5) it is possible to restrict oneself, without loss of generality, to the class of sequential approximate solutions (solution sequences).

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